

# Gaussian fluctuations of eigenvalues in the unitary ensemble

Deng Zhang

*Department of Mathematics, Shanghai Jiao Tong University,*

*Shanghai, 200240, China.*

*E-mail: zhangdeng@amss.ac.cn*

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## Abstract

This paper is devoted to the limit distribution of the  $k$ -th eigenvalue  $x_k$  in the unitary ensemble with the Freud-type potential, namely,

$$V_n(x) = \frac{1}{n}Q(c_n x + d_n),$$

where  $Q(x) = \sum_{k=0}^{2m} q_k x^k$ ,  $m \in \mathbb{N}^+$ ,  $c_n = \frac{\beta_n - \alpha_n}{2}$ ,  $d_n = \frac{\beta_n + \alpha_n}{2}$ , and  $\alpha_n, \beta_n$  are the  $n$ -th Mhasker-Rakhmanov-Saff numbers. With suitable scaling, we prove the central limit theorems for  $x_k$  in the bulk and edge cases respectively, as  $k$  tends to infinity with  $n$ . Moreover, multi-dimensional central limit theorems are also given.

These results generalize the works in [11] and [19]. In contrast to the latter, we employ here the Riemann-Hilbert method, developed in [5], to derive the asymptotic estimates of the Christoffel-Darboux kernels.

**Keywords:** Central limit theorem, the Costin-Lebowitz-Soshnikov theorem, Eigenvalues, Riemann-Hilbert problem, Unitary ensemble.

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## 1 Introduction and main results

We consider the unitary ensemble with the probability distribution on  $n \times n$  Hermitian matrices  $\mathcal{H}_n$ ,

$$\mathbb{P}_n(dH) = C_n e^{-n \text{Tr} V(H)} dH, \quad H \in \mathcal{H}_n. \quad (1.1)$$

where  $C_n$  is a normalization constant,  $V(x)$  is a real analytic potential with  $\frac{V(x)}{\log(x^2+1)} \rightarrow \infty$ , as  $|x| \rightarrow \infty$ , and  $dH = \prod_{1 \leq i < j \leq n} d\text{Re} H_{ij} d\text{Im} H_{ij} \prod_{i=1}^n dH_{ii}$ .

This probability distribution naturally induces a probability density function of the  $n$  ordered real eigenvalues  $\{x_i\}_{i=1}^n$ ,  $x_1 < \dots < x_n$ , given by

$$\mathcal{R}_{n,n}(x_1, \dots, x_n) = \frac{1}{Z_n} \prod_{1 \leq i < j \leq n} |x_i - x_j|^2 e^{-n[\sum_{i=1}^n V(x_i)]}, \quad (1.2)$$

where  $Z_n$  is a normalization constant (cf. e.g. (5.24) in [2]).

In the study of the random matrices theory, most interests center around the global and local limit behavior of eigenvalues. One remarkable global behavior is related with the equilibrium measure (see e.g. [2], [12]). As regards the local behavior, the fluctuation of the  $k$ -th eigenvalue  $x_k$  in the unitary ensemble is extensively studied in the literature and exhibits the universal behavior. When  $k$  is fixed, the fluctuation follows the celebrated Tracy-Widom distribution, which was first discovered in the GUE case [18] and later extended in [3] to the unitary ensembles with the Freud-type potential. On the other hand, when  $k$  is not fixed and tends to infinity with  $n$ , the fluctuation of  $x_k$  follows the Gaussian distribution. This work was first studied in the GUE case [11] and later generalized to various ensembles. See e.g. [13] for the GOE and GSE ensembles, and see [16] for the complex covariance matrices. We also refer to [17] for the non-Gaussian Wigner matrices.

Inspired by the universal phenomena, it is natural and interesting to study in the general unitary ensemble the fluctuation of the  $k$ -th eigenvalue  $x_k$ , where  $k$  tends to infinity with  $n$ . Here the potential of particular interest is of Freud type, namely,

$$V(x) = V_n(x) = \frac{1}{n} Q(c_n x + d_n) \quad (1.3)$$

where  $Q(x) = \sum_{k=0}^{2m} q_k x^k$ ,  $m \in \mathbb{N}^+$ ,  $c_n = \frac{\beta_n - \alpha_n}{2}$ ,  $d_n = \frac{\beta_n + \alpha_n}{2}$ , and  $\alpha_n, \beta_n$  are the  $n$ -th Mhasker-Rakhmanov-Saff numbers (see Section 2 for details). The Freud-type potential is a natural extension of the monomial type, i.e.

$$V(x) = q_{2m} x^{2m}, \quad m \geq 1, \quad \text{and} \quad q_{2m} = \frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(\frac{2m+1}{2})}. \quad (1.4)$$

In particular, it includes the case that  $V(x) = 2x^2$  ( $m = 1$ ) corresponding to the classical Gaussian Unitary Ensemble (GUE). Therefore, the results presented in this article generalize those in [11] and [19].

Before we show the main results, let us first recall the equilibrium measure  $\mu_{V_n}$ , which is the unique minimizer of the variation problem

$$\inf_{\mu \in \mathcal{M}_1(\mathbb{R})} I_{V_n}(\mu). \quad (1.5)$$

Here  $\mathcal{M}_1(\mathbb{R}) = \{\mu : \int_{\mathbb{R}} d\mu = 1\}$ , and

$$I_{V_n}(\mu) = \iint \log |s - t|^{-1} d\mu(s) d\mu(t) + \int V_n(t) d\mu(t). \quad (1.6)$$

It is known that (cf. [5]) for the Freud-type potential,  $\mu_{V_n}$  is absolutely continuous to the Lebesgue measure with the density function  $\rho_{V_n}$ .  $\rho_{V_n}$  is also uniquely determined by the Euler-Lagrange equations below (cf. (4.18), (4.19) in [5])

$$2 \int \log |x - s| \rho_{V_n}(s) ds - V_n(x) = l_n, \quad x \in [-1, 1] \quad (1.7)$$

$$2 \int \log |x - s| \rho_{V_n}(s) ds - V_n(x) \leq l_n, \quad x \in \mathbb{R} \setminus [-1, 1]. \quad (1.8)$$

The main results are given below in Theorem 1.1 and Theorem 1.2, corresponding to the bulk and edge cases respectively.

**Theorem 1.1** (*Bulk case.*)

Consider the unitary ensemble (1.1) with the Freud-type potential (1.3).

(i). Set  $G(s) = \int_{-1}^s \rho_{V_n}(x) dx$ ,  $-1 \leq s \leq 1$ , and  $t = t(k, n) = G^{-1}(k/n)$ , where  $k = k(n) \in [cn, (1-c)n]$ ,  $c \in (0, \frac{1}{2})$ . Then, as  $n \rightarrow \infty$ ,

$$\frac{x_k - t}{\frac{\sqrt{\log n}}{\sqrt{2\pi^2 n \rho_{V_n}(t)}}} \rightarrow N(0, 1)$$

in distribution.

(ii). Let  $\{x_{k_i}\}_{i=1}^m$  be eigenvalues such that  $0 < k_i - k_{i+1} \sim n^{\theta_i}$ ,  $0 < \theta_i \leq 1$ , and  $k_i \in [c_i n, (1-c_i)n]$ ,  $c_i \in (0, \frac{1}{2})$ . Define  $s_i = s_i(k_i, n) = G^{-1}(\frac{k_i}{n})$  and set

$$X_i = \frac{x_{k_i} - s_i}{\frac{\sqrt{\log n}}{\sqrt{2\pi^2 n \rho_{V_n}(s_i)}}}, \quad i = 1, \dots, m$$

Then as  $n \rightarrow \infty$ ,

$$\mathbb{P}_n[X_1 \leq \xi_1, \dots, X_m \leq \xi_m] \rightarrow \Phi_\Lambda(\xi_1, \dots, \xi_m).$$

Here  $\Phi_\Lambda$  is the  $m$ -dimensional Normal distribution function with mean zero and correlation matrix  $\Lambda$ ,  $\Lambda_{i,i} = 1$ ,  $1 \leq i \leq m$ , and  $\Lambda_{i,j} = 1 - \max_{i \leq k < j \leq m} \theta_k$ ,  $1 \leq i < j \leq m$ .

**Theorem 1.2** (*Edge case.*)

Consider the unitary ensemble (1.1) with the Freud-type potential (1.3).

(i). Let  $k$  be such that  $k \rightarrow \infty$  and  $\frac{k}{n} \rightarrow 0$  as  $n \rightarrow \infty$ . Then as  $n \rightarrow \infty$ ,

$$\frac{x_{n-k} - [1 - (\frac{k}{a_2 n})]^{\frac{2}{3}}}{a_1 \frac{\sqrt{\log k}}{n^{\frac{2}{3}} k^{\frac{1}{3}}}} \rightarrow N(0, 1) \quad (1.9)$$

in distribution, where  $a_1 = \frac{\sqrt{2}}{3\pi a_2^{\frac{2}{3}}}$ ,  $a_2 = \frac{2\sqrt{2}}{3\pi} \sum_{i=0}^{m-1} \frac{A_{m-1-i}}{A_m}$ ,  $A_j = \prod_{i=1}^j \frac{2i-1}{2i}$ ,  $1 \leq j \leq m$ ,  $A_0 = 1$ .

(ii). Let  $\{x_{k_i}\}_{i=1}^m$  be eigenvalues such that  $k_1 \sim n^\gamma$ ,  $0 < \gamma < 1$ , and  $0 < k_{i+1} - k_i \sim n^{\theta_i}$ ,  $0 < \theta_i < \gamma$ . Set

$$X_i = \frac{x_{n-k_i} - (1 - \frac{k_i}{a_2 n})^{\frac{2}{3}}}{a_1 \frac{\sqrt{\log k_i}}{n^{\frac{2}{3}} k_i^{\frac{1}{3}}}}, \quad i = 1, \dots, m. \quad (1.10)$$

Then as  $n \rightarrow \infty$ ,

$$\mathbb{P}_n[X_1 \leq \xi_1, \dots, X_m \leq \xi_m] \rightarrow \Phi_\Lambda(\xi_1, \dots, \xi_m),$$

where  $\Lambda$  is as in Theorem 1.1 but with  $\Lambda_{i,j} = 1 - \frac{1}{\gamma} \max_{i \leq k < j \leq m} \theta_k$ .

**Remark 1.3** Similar results can be proved for the uniform convex potential  $V$ , i.e.  $\inf_{\mathbb{R}} V'' \geq c > 0$  for some positive constant  $c$ . Since the strategy, based on the Riemann-Hilbert method developed in [4], is analogous to that in this paper, we omit the proof here and refer the interested reader to [20] for more details.

The Gaussian behavior in the two theorems above follows indeed from the celebrated Costin-Lebowitz-Soshnikov central limit theorem (cf. [1], [14] and [15]), thanks to the determinantal structure of the unitary ensemble. The proof thus relies on the asymptotic estimates of the expectation  $\mathbb{E}(\#I_n)$  and the variance  $\text{Var}(\#I_n)$ , where  $\#I_n$  denotes the number of eigenvalues in the interval  $I_n$ . As these two probabilistic quantities can be expressed in terms of the Christoffel-Darboux kernels  $\mathcal{K}_n(x, x)$  and  $\mathcal{K}_n(x, y)$  (see (4.2) and (4.5) below), the crucial technical points consequently lie in the asymptotic estimates of these kernels.

The case of monomial potential  $V$  is studied in the recent work [19]. However, more difficulties arise for the Freud-type potential. One is due to the lack of the symmetry  $\mathcal{K}_n(x, x) = \mathcal{K}_n(-x, -x)$ , then in order to obtain the estimates of  $\mathbb{E}(\#I_n)$ , one has to derive the asymptotic estimates of  $\mathcal{K}_n(x, x)$  in the whole real line, i.e.  $x \in \mathbb{R}$ , not just in the interval  $(-1, 1)$  (cf. e.g. (4.2) in [9]). Moreover, since the expression of  $\rho_{V_n}$  is much more complicated, the arguments in [19] and [11] are not applicable here, e.g. we can not derive the formula in Lemma 4.1 (2) in [19].

In order to remedy these difficulties, inspired by the works [4], [5] and [16], we employ here the Riemann-Hilbert method to derive the asymptotic estimates of  $\mathcal{K}_n(x, x)$  and  $\mathcal{K}_n(x, y)$ , which give us the asymptotics behavior of  $\mathbb{E}(\#I_n)$ ,  $\text{Var}(\#I_n)$  and then lead to the main results. As mentioned above, the results obtained in this paper consequently generalize the works [11] and [19].

This article is structured as follows. In Section 2 we briefly recall the Riemann-Hilbert method, which is applied in Section 3 to derive the crucial asymptotics of  $\mathcal{K}_n(x, x)$  and  $\mathcal{K}_n(x, y)$ . Section 4 and Section 5 are devoted to the proofs of the main results in the bulk and edge cases respectively. For simplicity of exposition, some technical details are postponed to the Appendix.

Throughout this article,  $\#I$  denotes the number of eigenvalues in the interval  $I \subset \mathbb{R}$ ,  $f = \mathcal{O}(g)$  means that  $|f/g|$  stays bounded, and  $C$  and  $c$  are constants which may change from one line to another.

## 2 Riemann-Hilbert method

Let us start with the Freud-type potential  $V_n(x)$  and the equilibrium density function  $\rho_{V_n}$ .

Let  $Q(x) = \sum_{k=0}^{2m} q_k x^k$  with  $q_{2m} = \frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(\frac{2m+1}{2})}$ ,  $m \geq 1$ . Define the  $n$ -th Mhasker-Rakhmanov-

Saff numbers  $\alpha_n, \beta_n$ ,

$$\frac{1}{2\pi} \int_{\alpha_n}^{\beta_n} \frac{Q'(t)(t - \alpha_n)}{\sqrt{(\beta_n - t)(t - \alpha_n)}} dt = n \quad (2.1)$$

$$\frac{1}{2\pi} \int_{\alpha_n}^{\beta_n} \frac{Q'(t)(\beta_n - t)}{\sqrt{(\beta_n - t)(t - \alpha_n)}} dt = -n. \quad (2.2)$$

It is known that  $\alpha_n$  and  $\beta_n$  exist for  $n$  large enough and they can be expressed in a power series in  $n^{-\frac{1}{2m}}$  (cf. Proposition 5.2 in [5]). In fact, the MRS numbers are chosen to insure that the support of  $\rho_{V_n}$  is  $[-1, 1]$ .

Set

$$c_n = \frac{\beta_n - \alpha_n}{2}, \quad d_n = \frac{\beta_n + \alpha_n}{2},$$

and

$$V_n(x) = \frac{1}{n} Q(c_n x + d_n).$$

We have that (cf. (5.17) and (5.18) in [5]),

$$V_n = \sum_{k=0}^{2m} v_{n,k} x^k \in \mathbb{P}_{2m}^+,$$

where

$$v_{n,2m} = \frac{1}{mA_m} + \mathcal{O}(n^{-\frac{1}{m}}), \quad v_{n,k} = \mathcal{O}(n^{\frac{k}{2m}-1}), \quad (2.3)$$

$$0 \leq k \leq 2m-1, \text{ and } A_m = \prod_{j=1}^m \frac{2j-1}{2j}.$$

**Example.** When  $Q$  is the monomial weight (1.4),  $\beta_n = -\alpha_n = n^{\frac{1}{2m}}$ , hence  $c_n = n^{\frac{1}{2m}}$ ,  $d_n = 0$  and  $V_n(x) \equiv Q(x)$ , which implies that the Freud-type weight is a natural generalization of the monomial weight.

For the corresponding equilibrium density function  $\rho_{V_n}$ , we have

**Theorem 2.1** ([5]) *There exists  $N > 0$ , such that for all  $n \geq N$ ,*

$$\rho_{V_n}(x) = \frac{1}{2\pi} \sqrt{1-x^2} h_n(x) \chi_{[-1,1]}(x), \quad (2.4)$$

where

$$h_n(x) = \sum_{k=0}^{2m-2} h_{n,k} x^k, \quad h_{n,k} = \sum_{j=0}^{\lfloor \frac{2m-2-k}{2} \rfloor} A_j(k+2+2j) v_{n,k+2+2j}. \quad (2.5)$$

Furthermore, there exists a constant  $h_0 > 0$  such that  $h_n(x) > h_0$  for all  $n \geq N$  and  $x \in \mathbb{R}$ .

From now on, we assume that  $n$  is sufficiently large for Theorem 2.1 to hold.

Set

$$F_n(x) = \left| \int_x^1 \frac{1}{2\pi} \sqrt{1-y^2} h_n(y) dy \right|, \quad (2.6)$$

and

$$\tilde{F}_n(x) = \left| \int_{-1}^x \frac{1}{2\pi} \sqrt{1-y^2} h_n(y) dy \right|. \quad (2.7)$$

Note that,  $F_n(x) = \int_x^1 \rho_{V_n}(y) dy$  for  $x \in (-1, 1)$ , and  $F_n(x) = \tilde{F}_n(-x)$  if  $Q(x) = Q(-x)$ .

Define the  $j^{th}$  orthogonal polynomials  $p_j(x)$  and the Christoffel-Darboux kernel  $K_j(x, y)$  with respect to the weight  $e^{-Q(x)}$ , i.e.

$$p_j(x) = \gamma_j x^j + \dots, \quad \gamma_j > 0, \quad j = 0, 1, \dots, \quad (2.8)$$

$$\int p_i(x) p_j(x) e^{-Q(x)} dx = \delta_{ij}, \quad i, j = 0, 1, 2, \dots,$$

$$K_j(x, y) = \sum_{i=0}^{j-1} p_i(x) p_i(y) e^{-\frac{Q(x)+Q(y)}{2}}, \quad j = 1, 2, \dots \quad (2.9)$$

Similarly, define  $p_j(x; n)$  and  $\mathcal{K}_j(x, y)$  with respect to the scaled weight  $e^{-nV_n(x)} (= e^{-Q(c_n x + d_n)})$ , i.e.,

$$p_j(x; n) = \gamma_j^{(n)} \pi_j(x; n), \quad (2.10)$$

where  $\gamma_j^{(n)} > 0$ ,  $\pi_j(x; n)$  is the monic polynomial,

$$\int p_i(x; n) p_j(x; n) e^{-nV_n(x)} dx = \delta_{ij}, \quad i, j = 0, 1, 2, \dots,$$

and

$$\mathcal{K}_j(x, y) = \sum_{i=0}^{j-1} p_i(x; n) p_i(y; n) e^{-n \frac{V_n(x) + V_n(y)}{2}}, \quad j = 1, 2, \dots \quad (2.11)$$

It is easy to verify that

$$p_i(x; n) = \sqrt{c_n} p_i(c_n x + d_n) \quad (2.12)$$

$$\gamma_i^{(n)} = c_n^{i+\frac{1}{2}} \gamma_i, \quad i = 0, 1, 2, \dots \quad (2.13)$$

$$\mathcal{K}_n(x, y) = c_n K_n(c_n x + d_n, c_n y + d_n) \quad (2.14)$$

In the following we briefly recall the Riemann-Hilbert method, which was introduced by Deift and Zhou in [8] and further developed in [7], [4], [5] and [6]. For more details of the following part we refer to [5].

Let  $U : \mathbb{C}/\mathbb{R} \rightarrow C^{2 \times 2}$  be an analytic matrix-valued function, which solves the following Riemann-Hilbert problem,

$$U_+(s) = U_-(s) \begin{pmatrix} 1 & e^{-nV_n(s)} \\ & 1 \end{pmatrix}, \quad s \in \mathbb{R}$$

$$U(z) \begin{pmatrix} z^{-n} & \\ & z^n \end{pmatrix} = I + \mathcal{O}\left(\frac{1}{|z|}\right), \text{ as } |z| \rightarrow \infty.$$

The fundamental relation between the Riemann-Hilbert problem and the orthogonal polynomial, observed by Fokas, Its and Kitaev [10], is that

$$U_{11}(z) = \frac{1}{\gamma_n^{(n)}} p_n(z; n), \quad U_{21}(z) = -2\pi i \gamma_{n-1}^{(n)} p_{n-1}(z; n). \quad (2.15)$$

Set

$$g_n(z) = \int_{-1}^1 \psi_n(t) \log(z-t) dt, \quad z \in \mathbb{C}/(-\infty, 1],$$

where

$$\psi_n(z) = \frac{1}{2\pi} (1-z)^{\frac{1}{2}} (1+z)^{\frac{1}{2}} h_n(z), \quad z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)) \quad (2.16)$$

with the analytic branch chosen by  $\arg(1-x) = \arg(1+x) = 0$ ,  $x \in (-1, 1)$ . Let

$$\xi_n(z) = -2\pi i \int_1^z \psi_n(y) dy, \quad z \in \mathbb{C}/(-\infty, -1] \cup [1, \infty),$$

It holds that (cf.(8.29) in [5])

$$g_n(z) = \frac{1}{2} (V_n(z) + l_n + \xi_n(z)), \quad z \in \mathbb{C}^+. \quad (2.17)$$

By the Pauli matrix  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , we define the matrix transformations,

$$T(z) = e^{-n \frac{l_n}{2} \sigma_3} U(z) e^{-n(g_n(z) - \frac{l_n}{2}) \sigma_3}, \quad z \in \mathbb{C}/\mathbb{R}. \quad (2.18)$$

and

$$S(z) = \begin{cases} T(z), & \text{for } z \text{ outside the lens-shaped region;} \\ T(z) \begin{pmatrix} 1 & 0 \\ -e^{-n\xi_n} & 1 \end{pmatrix}, & \text{in the upper lens region;} \\ T(z) \begin{pmatrix} 1 & 0 \\ e^{n\xi_n} & 1 \end{pmatrix}, & \text{in the lower lens region,} \end{cases} \quad (2.19)$$

with the lens regions defined as in [5], fig. 6.1.

Next we introduce the delicate paramatrices  $P_n$  in  $U_{\pm 1}$ , which are the small balls centered on  $\pm 1$  respectively with the radius  $\delta$  sufficient small. Let us first define  $f_n$  and  $\tilde{f}_n$  in  $U_1$  and  $U_{-1}$  respectively,

$$(-f_n(z))^{\frac{3}{2}} = -n \frac{3\pi}{2} \int_1^z \psi_n(y) dy, \quad z \in U_1/[1, \infty), \quad (2.20)$$

and

$$(\tilde{f}_n(z))^{\frac{3}{2}} = n \frac{3\pi}{2} \int_{-1}^z \psi_n(y) dy, \quad z \in U_{-1}/(-\infty, -1]. \quad (2.21)$$

It holds that (cf.(7.14), (7.21), (7.38), (7.36) and (7.37) in [5]),

$$\frac{2}{3}(f_n(z))^{\frac{2}{3}} = n\varphi_n(z), \text{ or } , f_n(z) = n^{\frac{2}{3}}(z-1)(\widehat{\phi}_n(z))^{\frac{2}{3}}, \quad (2.22)$$

and

$$\frac{2}{3}(-\widetilde{f}_n(z))^{\frac{2}{3}} = n\widetilde{\varphi}_n(z), \text{ or } , \widetilde{f}_n(z) = n^{\frac{2}{3}}(z+1)(\widehat{\phi}_n(z))^{\frac{2}{3}}, \quad (2.23)$$

where

$$\varphi_n(z) = \begin{cases} -\frac{1}{2}\xi_n(z) = \pi i \int_1^z \psi_n(y)dy, & z \in \mathbb{C}^+; \\ \frac{1}{2}\xi_n(z) = -\pi i \int_1^z \psi_n(y)dy, & z \in \mathbb{C}^-; \end{cases} \quad (2.24)$$

$$\widetilde{\varphi}_n(z) = \begin{cases} \varphi_n(z) + \pi i = \pi i \int_{-1}^z \psi_n(y)dy, & z \in \mathbb{C}^+; \\ \varphi_n(z) - \pi i = -\pi i \int_{-1}^z \psi_n(y)dy, & z \in \mathbb{C}^-; \end{cases} \quad (2.25)$$

and  $\widehat{\phi}_n, \widehat{\phi}_n$  are analytic functions in  $U_1$  and  $U_{-1}$  respectively.

The paramatrices  $P_n$  in  $U_{\pm 1}$  are defined below.

(i). For  $z \in U_1/f_n^{-1}(\gamma_\sigma)$  with the contour  $\gamma_\sigma$  defined as in [5], fig. 7.1,

$$P_n := E_n \Psi^\sigma(f_n) e^{n\varphi_n \sigma_3}, \quad (2.26)$$

where  $E_n = \sqrt{\pi} e^{\frac{\pi i}{6}} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \begin{pmatrix} H_n & \\ & H_n^{-1} \end{pmatrix}$ ,  $H_n = f_n^{\frac{1}{4}} a^{-1}$ , and

$$\Psi^\sigma(z) = \begin{cases} AI(z) e^{-\frac{\pi i}{6} \sigma_3}, & z \in I : 0 < \arg z < \frac{2\pi}{3}; \\ AI(z) e^{-\frac{\pi i}{6} \sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & z \in II : \frac{2\pi}{3} < \arg z < \pi; \\ \widetilde{AI}(z) e^{-\frac{\pi i}{6} \sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in III : -\pi < \arg z < -\frac{2\pi}{3}; \\ \widetilde{AI}(z) e^{-\frac{\pi i}{6} \sigma_3}, & z \in IV : -\frac{2\pi}{3} < \arg z < 0. \end{cases} \quad (2.27)$$

Here  $AI(z), \widetilde{AI}(z)$  denote the matrices  $\begin{pmatrix} Ai(z) & Ai(\omega^2 z) \\ Ai'(z) & \omega^2 Ai'(\omega^2 z) \end{pmatrix}$  and  $\begin{pmatrix} Ai(z) & -\omega^2 Ai(\omega z) \\ Ai'(z) & -Ai'(\omega z) \end{pmatrix}$  respectively,  $\omega = e^{\frac{2\pi i}{3}}$ , and  $Ai$  means the Airy function, which is uniquely determined by  $Ai''(z) = zAi(z)$  with  $\lim_{x \rightarrow \infty} \sqrt{4\pi} x^{\frac{1}{4}} e^{\frac{2}{3}x^{\frac{3}{2}}} Ai(x) = 1$ .

(ii). For  $z \in U_{-1}/\widetilde{f}_n^{-1}(\widetilde{\gamma}_\sigma)$  with the contour  $\widetilde{\gamma}_\sigma$  defined as in [5], fig. 7.3,

$$P_n := \widetilde{E}_n \widetilde{\Psi}^\sigma(\widetilde{f}_n) e^{n\widetilde{\varphi}_n \sigma_3}, \quad (2.28)$$

where  $\widetilde{\Psi}^\sigma(z) = \sigma_3 \Psi^\sigma(-z) \sigma_3$ ,  $\widetilde{E}_n = \sqrt{\pi} e^{\frac{\pi i}{6}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \widetilde{H}_n & \\ & \widetilde{H}_n^{-1} \end{pmatrix}$ , and  $\widetilde{H}_n = (-\widetilde{f}_n)^{\frac{1}{4}} a$ .

Finally, set

$$R = \begin{cases} SP_n^{-1}, & \text{for } z \in U_1 \cup U_{-1}; \\ SN^{-1}, & \text{otherwise,} \end{cases} \quad (2.29)$$

where

$$N = \frac{1}{2} \begin{pmatrix} a + a^{-1} & i(a^{-1} - a) \\ i(a - a^{-1}) & a + a^{-1} \end{pmatrix}, \quad (2.30)$$



and

$$a(z) = \left(\frac{z-1}{z+1}\right)^{\frac{1}{4}}, \quad z \in \mathbb{C}/[-1, 1]. \quad (2.31)$$

with the analytic branch chosen by  $\arg(x-1) = \arg(x+1) = 0$ , for  $x > 1$ . It is known that  $R$  has the following asymptotic expansions (cf. (7.64) in [5], and (3.6), (3.7) in [3]),

$$R(z) = I + \frac{1}{n} \sum_{k=0}^{\infty} r_k(z) n^{-\frac{k}{2m}}, \quad (2.32)$$

$$\frac{d}{dz} R = \frac{1}{n} \sum_{k=0}^{\infty} \frac{d}{dz} r_k(z) n^{-\frac{k}{2m}}, \quad (2.33)$$

where  $r_k(z)$ ,  $\frac{d}{dz} r_k(z)$ ,  $0 \leq k < \infty$ , are bounded functions and analytic in the complement of the set  $\partial U_1 \cup \partial U_{-1}$ , and these expansions are uniform for  $z \in \mathbb{C}/\widehat{\Sigma}_R$  with  $\widehat{\Sigma}_R$  defined as in [5] fig.7.6.

**Remark 2.2** *If  $z = x \in \mathbb{R}$ , we always take the limiting expressions as  $z$  is approaching from the upper half-plane. So, if  $x > 1$ ,  $\psi_n(x)$  means that  $\lim_{\epsilon \rightarrow 0^+} \psi_n(x + i\epsilon)$ .*

### 3 Asymptotics of the kernels

This section is devoted to the asymptotic estimates of the Christoffel-Darboux kernels  $\mathcal{K}_n(x, x)$  and  $\mathcal{K}_n(x, y)$ .

Let us start with  $\mathcal{K}_n(x, x)$ , which is crucial for the estimates of the expectation in the next sections.

**Lemma 3.1** *Take any sufficient small  $\delta > 0$ , the following holds:*

(i). *For  $x \in (-1 + \delta, 1 - \delta)$ ,*

$$\mathcal{K}_n(x, x) = n\rho_{V_n}(x) + \mathcal{O}(1). \quad (3.1)$$

(ii). *For  $x \in (1 - \delta, 1 + \delta)$ ,*

$$\begin{aligned} \mathcal{K}_n(x, x) &= \left[ \frac{1}{4} \frac{f'_n(x)}{f_n(x)} - \frac{a'(x)}{a(x)} \right] 2Ai(f_n(x))Ai'(f_n(x)) \\ &\quad + f'_n(x) [(Ai')^2(f_n(x)) - f_n(x)Ai^2(f_n(x))] + \mathcal{O}(n^{-\frac{5}{6}}). \end{aligned} \quad (3.2)$$

(iii). *For  $x \in (-1 - \delta, -1 + \delta)$ ,*

$$\begin{aligned} \mathcal{K}_n(x, x) &= - \left[ \frac{1}{4} \frac{\widetilde{f}'_n(x)}{\widetilde{f}_n(x)} + \frac{a'(x)}{a(x)} \right] 2Ai(-\widetilde{f}_n(x))Ai'(-\widetilde{f}_n(x)) \\ &\quad + \widetilde{f}'_n(x) [(Ai')^2(-\widetilde{f}_n(x)) + \widetilde{f}_n(x)Ai^2(-\widetilde{f}_n(x))] + \mathcal{O}(n^{-\frac{5}{6}}). \end{aligned} \quad (3.3)$$

(iv). *For  $x \in (\infty, \infty)/(-1 - \delta, 1 + \delta)$ ,*

$$\mathcal{K}_n(x, x) = \frac{1}{4\pi} \frac{1}{(x-1)(x+1)} e^{-2n\varphi_n(x)} + \mathcal{O}(n^{-1}). \quad (3.4)$$

**Remark 3.2** Analogous results are proved in [16] for the complex covariance matrices. However, the method given below, inspired by [3], is quite different than that in [16] and gives us the main order directly. Some technical details are postponed to the Appendix.

**Proof.** The starting point is the formula

$$2\pi i(x-y)\mathcal{K}_n(x,y) = (1,0)U(x)^T U(y)^{-T} (0,1)^T e^{-n\frac{V_n(x)+V_n(y)}{2}}, \quad (3.5)$$

which follows straightforwardly from the Christoffel-Darboux formula (cf. e.g. (3.48) in [2]) and the relation (2.15). We emphasize that, (3.5) allows to apply the Riemann-Hilbert method, developed to analyze the asymptotics of solution  $U$  to the Riemann-Hilbert problem, to obtain the asymptotic estimates of  $\mathcal{K}_n(x,y)$ .

(i). For  $x, y \in (-1+\delta, 1-\delta)$ , by (2.18) and (2.19),

$$U = e^{\frac{n}{2}l_n\sigma_3} S \begin{pmatrix} 1 & 0 \\ e^{-n\xi_n} & 1 \end{pmatrix} e^{n(g_n - \frac{l_n}{2})\sigma_3}. \quad (3.6)$$

hence, by (3.5), (2.17) and (2.24), direct computations show that

$$2\pi i(x-y)\mathcal{K}_n(x,y) = (e^{-n\varphi_n(x)}, e^{n\varphi_n(x)})S(x)^T S(y)^{-T} (-e^{n\varphi_n(y)}, e^{-n\varphi_n(y)})^T. \quad (3.7)$$

In order to obtain the main order, we note that  $S^T(x) = S^T(y) + (x-y)\Delta_S(x,y)$ , where

$$\Delta_S(x,y) = \int_0^1 (S^T)'(y+t(x-y))dt, \quad (3.8)$$

then

$$S^T(x)S^{-T}(y) = Id + (x-y)\Delta_S(x,y)S^{-T}(y). \quad (3.9)$$

Hence plugging (3.9) into (3.7), together with  $\varphi_n(x) = -\pi i F_n(x)$ ,  $x \in (-1,1)$ , we obtain that

$$2\pi i(x-y)\mathcal{K}_n(x,y) = -2i \sin[n\pi(F_n(x) - F_n(y))] + (x-y)I_1(x,y), \quad (3.10)$$

and

$$I_1(x,y) := (e^{-n\varphi_n(x)}, e^{n\varphi_n(x)})[\Delta_S(x,y)S^{-T}(y)](-e^{n\varphi_n(y)}, e^{-n\varphi_n(y)})^T,$$

Now, letting  $x = y$  in (3.10) we come to

$$\begin{aligned} & 2\pi i\mathcal{K}_n(x,x) \\ &= 2n\pi i\rho_{V_n}(x) + (e^{-n\varphi_n(x)}, e^{n\varphi_n(x)}) [(S^T)'(x)S^{-T}(x)] (-e^{n\varphi_n(x)}, e^{-n\varphi_n(x)})^T. \end{aligned}$$

Since by (2.29), (2.30), (2.32) and (2.33),  $S(x)$  and  $S'(x)$  are uniformly bounded for  $x \in [-1+\delta, 1-\delta]$ , we consequently obtain (3.1).

(ii). For  $x, y \in (1-\delta, 1)$ , or,  $x, y \in (1, 1+\delta)$ , similar calculations show that,

$$\begin{aligned} 2\pi i(x-y)\mathcal{K}_n(x,y) &= e^{-\frac{\pi i}{3}}(1,0)[AI(f_n(x))]^T E_n^T(x)R^T(x) \\ &\quad R^{-T}(y)E_n^{-T}(y)[AI(f_n(y))]^{-T}(0,1)^T. \end{aligned} \quad (3.11)$$

(see the Appendix for the proof.)

To obtain the main order, we apply (3.9) with  $S$  replaced by  $R$  and derive

$$2\pi i(x-y)\mathcal{K}_n(x,y) = e^{-\frac{\pi i}{3}}(1,0)[AI(f_n(x))]^T E_n^T(x) E_n^{-T}(y)[AI(f_n(y))]^{-T}(0,1)^T \\ + (x-y)e^{-\frac{\pi i}{3}} I_2(x,y), \quad (3.12)$$

where

$$I_2(x,y) := (1,0)[AI(f_n(x))]^T E_n^T(x) \\ \Delta_R(x,y) R^{-T}(y) E_n^{-T}(y)[AI(f_n(y))]^{-T}(0,1)^T. \quad (3.13)$$

Then by the explicit expressions of  $AI$ ,  $E_n$  and the asymptotics (2.32) and (2.33), it is not difficult to deduce that  $I_2(x,y)$  is of order  $n^{-\frac{5}{6}}$ , which does not contribute the main order, and

$$2\pi i(x-y)\mathcal{K}_n(x,y) = (-2\pi i) \left[ -Ai(f_n(x))Ai'(f_n(y)) \frac{f_n^{\frac{1}{4}}(x)}{f_n^{\frac{1}{4}}(y)} \frac{a(y)}{a(x)} \right. \\ \left. + Ai'(f_n(x))Ai(f_n(y)) \frac{f_n^{\frac{1}{4}}(y)}{f_n^{\frac{1}{4}}(x)} \frac{a(x)}{a(y)} \right] + (x-y)\mathcal{O}(n^{-\frac{5}{6}}). \quad (3.14)$$

(See the Appendix for the proof.)

Therefore, in order to get  $\mathcal{K}_n(x,x)$ , we just need to take the Taylor expansion, which together with  $Ai''(x) = xAi(x)$  leads to

$$2\pi i(x-y)\mathcal{K}_n(x,y) \\ = -2\pi i(y-x) \left\{ \left[ \frac{1}{4} \frac{f'_n(x)}{f_n(x)} - \frac{a'(x)}{a(x)} \right] 2Ai(f_n(x))Ai'(f_n(x)) \right. \\ \left. - f_n(x)f'_n(x)Ai^2(f_n(x)) + f'_n(x)(Ai')^2(f_n(x)) \right\} + \mathcal{O}((y-x)^2) + (x-y)\mathcal{O}(n^{-\frac{5}{6}}),$$

thereby implying (3.2).

(iii). For  $x, y \in (-1-\delta, -1)$ , or,  $x, y \in (-1, -1+\delta)$ , the proof is analogous to the previous case. We first compute that

$$2\pi i(x-y)\mathcal{K}_n(x,y) = (-1)e^{-\frac{\pi i}{3}}(1,0)[\widetilde{AI}(-\widetilde{f}_n(x))]^T \sigma_3 \widetilde{E}_n^T(x) R^T(x) \\ R^{-T}(y) \widetilde{E}_n^{-T}(y) \sigma_3^{-1} [\widetilde{AI}(-\widetilde{f}_n(y))]^{-T}(0,1)^T. \quad (3.15)$$

(See the Appendix for the proof.)

Then, by (3.9) with  $S$  replaced by  $R$ ,

$$2\pi i(x-y)\mathcal{K}_n(x,y) \\ = (-1)e^{-\frac{\pi i}{3}}(1,0)[\widetilde{AI}(-\widetilde{f}_n(x))]^T \sigma_3 \widetilde{E}_n^T(x) \\ \widetilde{E}_n^{-T}(y) \sigma_3^{-1} [\widetilde{AI}(-\widetilde{f}_n(y))]^{-T}(0,1)^T - e^{-\frac{\pi i}{3}}(x-y)I_3(x,y), \quad (3.16)$$

where

$$I_3(x,y) := (1,0)[\widetilde{AI}(-\widetilde{f}_n(x))]^T \sigma_3 \widetilde{E}_n^T(x) \\ \Delta_R(x,y) R^{-T}(y) \widetilde{E}_n^{-T}(y) \sigma_3^{-1} [\widetilde{AI}(-\widetilde{f}_n(y))]^{-T}(0,1)^T. \quad (3.17)$$

Hence, using the expressions of  $\widetilde{AI}$ ,  $\widetilde{f}_n$  and arguing as in the case (ii), we deduce that

$$\begin{aligned} & 2\pi i(x-y)\mathcal{K}_n(x,y) \\ &= (-2\pi i) \left[ Ai(-\widetilde{f}_n(x))Ai'(-\widetilde{f}_n(y)) \frac{\widetilde{f}_n(x)^{\frac{1}{4}} a(x)}{\widetilde{f}_n(y)^{\frac{1}{4}} a(y)} \right. \\ & \quad \left. - Ai'(-\widetilde{f}_n(x))Ai(-\widetilde{f}_n(y)) \frac{\widetilde{f}_n(y)^{\frac{1}{4}} a(y)}{\widetilde{f}_n(x)^{\frac{1}{4}} a(x)} \right] + (x-y)\mathcal{O}(n^{-\frac{5}{6}}). \end{aligned} \quad (3.18)$$

Consequently, taking the Taylor expansion yields that

$$\begin{aligned} & 2\pi i(x-y)\mathcal{K}_n(x,y) \\ &= -2\pi i(y-x) \left\{ - \left[ \frac{1}{4}\widetilde{f}_n^{-1}(x)\widetilde{f}_n'(x) + \frac{a'(x)}{a(x)} \right] 2Ai(-\widetilde{f}_n(x))Ai'(-\widetilde{f}_n(x)) \right. \\ & \quad \left. + \widetilde{f}_n'(x) \left[ \widetilde{f}_n(x)Ai^2(-\widetilde{f}_n(x)) + (Ai')^2(-\widetilde{f}_n(x)) \right] \right\} + \mathcal{O}(y-x)^2 + (x-y)\mathcal{O}(n^{-\frac{5}{6}}), \end{aligned}$$

which implies (3.3).

(iv). For  $x, y \in (-\infty, \infty)/(-1-\delta, 1+\delta)$ , by (2.18) and (2.19),

$$U = e^{n\frac{L_n}{2}\sigma_3} S e^{n(g_n - \frac{L_n}{2})\sigma_3},$$

which by (3.5) and (2.17) implies that

$$2\pi i(x-y)\mathcal{K}_n(x,y) = e^{-n(\varphi_n(x)+\varphi_n(y))}(1,0)S^T(x)S^{-T}(y)(0,1)^T.$$

Then by  $S = RN$ , and using (3.9) twice with  $S$  replaced by  $R$  and  $N$  respectively, we come to

$$\begin{aligned} 2\pi i(x-y)\mathcal{K}_n(x,y) &= (x-y)e^{-n(\varphi_n(x)+\varphi_n(y))}(1,0)\Delta_N(x,y)N^{-T}(y)(0,1)^T \\ & \quad + (x-y)I_4(x,y), \end{aligned} \quad (3.19)$$

where

$$I_4(x,y) := e^{-n(\varphi_n(x)+\varphi_n(y))}(1,0)N^T(x)\Delta_R(x,y)R^{-T}(y)N^{-T}(y)(0,1)^T.$$

Since  $N(x)$  and  $e^{-n\varphi_n(x)}$  are bounded for  $x \in \mathbb{R}/(-1-\delta, 1+\delta)$ , together with (2.32) and (2.33), we deduce that  $I_4(x,y) = \mathcal{O}(n^{-1})$ . Then,

$$\mathcal{K}_n(x,y) = \frac{1}{2\pi i} e^{-n(\varphi_n(x)+\varphi_n(y))}(1,0)\Delta_N(x,y)N^{-T}(y)(0,1)^T + \mathcal{O}(n^{-1}),$$

and

$$\mathcal{K}_n(x,x) = \frac{1}{2\pi i} e^{-2n\varphi_n(x)}(1,0)(N^T)'(x)N^{-T}(x)(0,1)^T + \mathcal{O}(n^{-1}),$$

which yields (3.4) by the explicit expressions of  $N$  and  $a$  in (2.30) and (2.31) respectively. The proof of Lemma 3.1 is consequently complete.  $\square$

We next consider  $\mathcal{K}_n(x,y)$ , which will be used for the estimates of variance in the next sections. In the bulk case we have

**Lemma 3.3** (i). Let  $t \in (-1, 1)$ , and set  $\Gamma_1^1 = \{(x, y) : t \leq x \leq t + \frac{1-t}{\log n}, t - \frac{1+t}{\log n} \leq y \leq t - \frac{1}{n}\}$ . Then, for  $(x, y) \in \Gamma_1^1$ ,

$$\mathcal{K}_n(x, y) = \frac{\sin[\pi n(F_n(y) - F_n(x))] + \mathcal{O}(\frac{1}{\log n})}{\pi(x - y)}, \quad (3.20)$$

where  $F_n(x)$  is defined as in (2.6).

(ii). Let  $\delta > 0$  be sufficiently small. For  $x, y \in [-1 + \delta, 1 - \delta]$ ,

$$\mathcal{K}_n^2(x, y) = \mathcal{O}(\frac{1}{(x - y)^2}). \quad (3.21)$$

(iii). Let  $t \in (-1, 1)$ . For  $(x, y) \in \{(x, y) : t \leq x \leq t + \frac{1}{n}, t - \frac{1}{n} \leq x \leq t\}$ ,

$$\mathcal{K}_n(x, y) = \mathcal{O}(n).$$

**Proof.** The formula (3.10) implies that

$$2\pi i(x - y)\mathcal{K}_n(x, y) = -2i \sin[n\pi(F_n(x) - F_n(y))] + \mathcal{O}(|x - y|), \quad (3.22)$$

which together with Lemma 4.1 leads to the assertions.  $\square$

Moreover, in the edge case we have

**Lemma 3.4** For  $x \in (1 - \delta, 1 + \delta)$ ,  $y \in (1 - \delta, 1)$  with  $\delta > 0$  sufficient small,

$$\begin{aligned} \mathcal{K}_n(x, y) = & \frac{1}{x - y} \left[ Ai(f_n(x))Ai'(f_n(y)) \frac{f_n^{\frac{1}{4}}(x)}{f_n^{\frac{1}{4}}(y)} \frac{a(y)}{a(x)} \right. \\ & \left. - Ai'(f_n(x))Ai(f_n(y)) \frac{f_n^{\frac{1}{4}}(y)}{f_n^{\frac{1}{4}}(x)} \frac{a(x)}{a(y)} \right] + \mathcal{O}(n^{-\frac{5}{6}}). \end{aligned} \quad (3.23)$$

**Proof.** Taking into account (3.14), we only need to prove (3.23) for  $x \in (1, 1 + \delta)$  and  $y \in (1 - \delta, 1)$ .

To this end, for  $x \in (1, 1 + \delta)$ , by (2.18), (2.19), (2.29) and (2.26),

$$U(x) = e^{n\frac{L_n}{2}\sigma_3} S(x) e^{n(g_n - \frac{L_n}{2})\sigma_3}, \quad (3.24)$$

where  $S(x) = R(x)E_n(x)[AI(f_n(x))]e^{-\frac{\pi i}{6}\sigma_3}e^{n\varphi_n\sigma_3}$ . Moreover, for  $y \in (1 - \delta, 1)$ ,

$$U(y) = e^{n\frac{L_n}{2}\sigma_3} S(y) \begin{pmatrix} 1 & 0 \\ e^{-n\xi_n} & 1 \end{pmatrix} e^{n(g_n - \frac{L_n}{2})\sigma_3} \quad (3.25)$$

with  $S(y) = R(y)E_n(y)[AI(f_n(y))]e^{-\frac{\pi i}{6}\sigma_3}e^{n\varphi_n\sigma_3}$ .

Hence, plugging (3.24) and (3.25) into (3.5), we come to

$$\begin{aligned} 2\pi i(x - y)\mathcal{K}_n(x, y) = & e^{-\frac{\pi i}{3}}(1, 0)[AI(f_n(x))]^T E_n^T(x) R^T(x) \\ & R^{-T}(y) E_n^{-T}(y)[AI(f_n(y))]^{-T}(0, 1)^T, \end{aligned} \quad (3.26)$$

which has the same expression as in (3.11). Therefore, similar arguments there imply (3.23) for  $x \in (1, 1 + \delta)$  and  $y \in (1 - \delta, 1)$ , thereby proving (3.23) as claimed.  $\square$

## 4 Bulk case.

The proof of Theorem 1.1 depends crucially on the estimates of the expectation and variance in Proposition 4.2 and Proposition 4.3 below respectively.

**Lemma 4.1** *Let  $\delta \in (0, 1)$  and  $N$  be defined as in Theorem 2.1. Then  $1/\rho_{V_n}$  and  $|\rho'_{V_n}|$  are uniformly bounded for all  $n \geq N$  and  $x \in [-1 + \delta, 1 - \delta]$ .*

(See the Appendix for the proof.)

**Proposition 4.2** *Let  $t = t(k, n)$  be defined as in theorem 1.1. Fix  $\xi \in \mathbb{R}$ , set  $a_n = \frac{\sqrt{\log n}}{\sqrt{2\pi^2 n \rho_{V_n}(t)}}$ ,  $t_n = t + a_n \xi$  and  $I_n = [t_n, \infty)$ . Then*

$$\mathbb{E}(\#I_n) = n - k - \frac{\sqrt{\log n}}{\sqrt{2\pi^2}} \xi + \mathcal{O}(1). \quad (4.1)$$

**Proof.** Since  $\sup_n |t(k, n)| = \sup_n |G^{-1}(\frac{k}{n})| < 1$ , Lemma 4.1 implies that  $1/\rho_V(t)$ ,  $n \in \mathbb{N}$ , are bounded and  $a_n = \mathcal{O}(\frac{\sqrt{\log n}}{n})$ . Note that, by lemma 3.1 (i) and (iv),

$$\begin{aligned} \mathbb{E}(\#I_n) &= \int_{t_n}^{\infty} \mathcal{K}_n(x, x) dx \\ &= \int_{t_n}^{1-\delta} n \rho_{V_n}(x) dx + \int_{1-\delta}^{1+\delta} \mathcal{K}_n(x, x) dx + \mathcal{O}(1). \end{aligned} \quad (4.2)$$

Then, as in [16] we deduce from Lemma 3.1 (ii) that

$$\int_{1-\delta}^{1+\delta} \mathcal{K}_n(x, x) dx = \int_{1-\delta}^1 n \rho_{V_n}(x) dx + \mathcal{O}(1). \quad (4.3)$$

Therefore, plugging (4.3) into (4.2), using  $\int_{-1}^1 \rho_{V_n}(x) dx = 1$  and taking the Taylor expansion, we derive that

$$\begin{aligned} \mathbb{E}(\#I_n) &= \int_{t_n}^1 n \rho_{V_n}(x) dx + \mathcal{O}(1) \\ &= n - n \int_{-1}^{t_n} \rho_{V_n}(x) dx + \mathcal{O}(1) \\ &= n - n \int_{-1}^t \rho_{V_n}(x) dx - n \int_t^{t+a_n \xi} \rho_{V_n}(x) dx + \mathcal{O}(1) \\ &= n - k - n \left[ \rho_{V_n}(t) a_n \xi + \frac{1}{2} \rho'_{V_n}(\eta) (a_n \xi)^2 \right] + \mathcal{O}(1) \end{aligned}$$

with  $\eta \in (t, t + a_n \xi)$ , which by the expression of  $a_n$  and Lemma 4.1 yields (4.1).  $\square$

**Proposition 4.3** *Let  $\{t_i\}_{i=1}^{\infty}$  be a sequence such that  $\sup_n |t_n| < 1$ . Set  $I_n = [t_n, \infty)$ ,  $n \in \mathbb{N}$ . Then*

$$\text{Var}(\#I_n) = \frac{1}{2\pi^2} \log n + \mathcal{O}(\log \log n). \quad (4.4)$$

**Proof.** Since the proof is similar as that in Lemma 3.2 in [19] (see also Lemma 2.3 in [11]), we just give a sketch of it. First note that

$$\begin{aligned} \text{Var}(\#I_n) &= \iint_{\Omega_n} \mathcal{K}_n^2(x, y) dx dy \\ &= \iint_{\Gamma} \mathcal{K}_n^2(x, y) dx dy + \iint_{\Omega_n/\Gamma} \mathcal{K}_n^2(x, y) dx dy, \end{aligned} \quad (4.5)$$

where  $\Omega_n = \{(x, y) : t_n \leq x < \infty, -\infty < y \leq t_n\}$  and  $\Gamma = \{(x, y) : t_n \leq x \leq 1 - \delta, -1 + \delta \leq y \leq t_n\}$ .

Using the asymptotic formulas of  $\mathcal{K}_n(x, y)$  in Lemma 3.3, we obtain that

$$\iint_{\Gamma} \mathcal{K}_n^2(x, y) dx dy = \frac{1}{2\pi^2} \log n + \mathcal{O}(\log \log n), \quad (4.6)$$

Here the main order  $\frac{1}{2\pi^2} \log n$  follows indeed from the integration on  $\Gamma_1^1$ , which is defined as in Lemma 3.3 (i).

As regards the remaining integration on  $\Omega_n/\Gamma$ , we first note that for  $(x, y) \in \Omega_n/\Gamma$ ,  $x - y \geq 2 - 2\delta > 0$ . Moreover, by (2.13) and the asymptotics of  $\gamma_n$  (cf. (2.11) in [5]),  $\frac{\gamma_{n-1}^{(n)}}{\gamma_n^{(n)}} = \frac{1}{2} + \mathcal{O}(\frac{1}{n^2})$ . It then follows from the Christoffel-Darboux identity that

$$\begin{aligned} \mathcal{K}_n^2(x, y) &\leq C \left\{ \left[ p_n(x; n) p_{n-1}(y; n) e^{-n \frac{V_n(x) + V_n(y)}{2}} \right]^2 \right. \\ &\quad \left. + \left[ p_n(y; n) p_{n-1}(x; n) e^{-n \frac{V_n(x) + V_n(y)}{2}} \right]^2 \right\}, \end{aligned}$$

which by Lemma 6.1 in the Appendix yields

$$\iint_{\Omega_n/\Gamma} \mathcal{K}_n^2(x, y) dx dy = \mathcal{O}(1). \quad (4.7)$$

Consequently, plugging (4.6) and (4.7) into (4.5) gives (4.4).  $\square$

**Proof of Theorem 1.1.** Take  $t, \xi, a_n$  and  $I_n$  as in Proposition 4.2. From Proposition 4.2 and Proposition 4.3 it follows that

$$\mathbb{P}\left(\frac{x_k - t}{a_n} < \xi\right) = \mathbb{P}(\#I_n \leq n - k) = \mathbb{P}\left(\frac{\#I_n - \mathbb{E}\#I_n}{\sqrt{\text{Var}(\#I_n)}} \leq \xi + o(1)\right).$$

Hence the Costin-Lebowitz-Soshnikov theorem (cf. [14]) yields the assertion (i).

The proof for (ii) is analogous. The limit normal behavior now follows from the Soshnikov central limit theorem in [15], and as in the proof of Proposition 4.3, the calculation for the correlation coefficient  $\Lambda_{i,j}$  is mainly based on the fact that for any given subset  $\Lambda \subset \Omega_n$ ,

$$\iint_{\Lambda} \mathcal{K}_n^2(x, y) dx dy = \iint_{\Lambda \cap \Gamma_1^1} \frac{1}{2\pi^2(x-y)^2} dx dy + \mathcal{O}(\log \log n).$$

For simplicity of exposition, we refer to [11] and [20] for more details.  $\square$

## 5 Edge case.

As in Section 4 let us start with the estimates of the expectation and variance in the edge case.

**Proposition 5.1** *Let  $I = [t, \infty)$  with  $t \rightarrow 1^-$ . Then*

$$\mathbb{E}(\#I) = \frac{2\sqrt{2}}{3\pi} \sum_{i=0}^{m-1} \frac{A_{m-1-i}}{A_m} n(1-t)^{\frac{3}{2}}(1+o(1)), \quad (5.1)$$

where  $A_j = \prod_{i=1}^j \frac{2i-1}{2i}$ ,  $1 \leq j \leq m$ , and  $A_0 = 1$ .

**Proof.** As in (4.2) and (4.3) we have that

$$\mathbb{E}(\#I) = \int_t^1 n\rho_{V_n}(x)dx + \mathcal{O}(1).$$

Since  $t \rightarrow 1^-$ , it then follows from the expression of  $\rho_{V_n}$  in (2.4) that

$$\mathbb{E}(\#I) = \frac{\sqrt{2}}{3\pi} h_n(1)n(1-t)^{\frac{3}{2}} + \mathcal{O}(1), \quad (5.2)$$

which by (6.10) in the Appendix yields (5.1).  $\square$

**Proposition 5.2** *Let  $t$  be such that  $t \rightarrow 1^-$  and  $n(1-t)^{\frac{3}{2}} \rightarrow \infty$ , and set  $I = [t, \infty)$ . Then*

$$\text{Var}(\#I) = \frac{1}{2\pi^2} \log \left[ n(1-t)^{\frac{3}{2}} \right] (1+o(1)). \quad (5.3)$$

**Proof.** Thanks to Lemma 3.4, the proof now follows analogously as that of Lemma 4 in [16], thus we just give a sketch of it below.

As in (4.5), we have that

$$\begin{aligned} \text{Var}(\#I) &= \iint_{\tilde{\Omega}_n} \mathcal{K}_n^2(x, y) dx dy \\ &= \iint_{\tilde{\Gamma}} \mathcal{K}_n^2(x, y) dx dy + \iint_{\tilde{\Omega}_n/\tilde{\Gamma}} \mathcal{K}_n^2(x, y) dx dy, \end{aligned} \quad (5.4)$$

where  $\tilde{\Omega}_n = \{(x, y) : t \leq x < \infty, -\infty < y \leq t\}$ ,  $\tilde{\Gamma} = \{(x, y) : t \leq x \leq t + \frac{1-t}{r_n}, t - \frac{1-t}{r_n} \leq y \leq t - \epsilon\}$  with  $\epsilon = \frac{1}{n\sqrt{1-t}}$  and  $\frac{1}{r_n} = \max\{\sqrt{1-t}, \frac{1}{\log[n(1-t)^{\frac{3}{2}}]}\}$ .

As in [16], we deduce from (3.23) that

$$\iint_{\tilde{\Gamma}} \mathcal{K}_n^2(x, y) dx dy = \frac{1}{2\pi^2} \log[n(1-t)^{\frac{3}{2}}] + \mathcal{O}(\log r_n), \quad (5.5)$$



which indeed contributes the main order in (5.3).

As regards the remaining integration on  $\tilde{\Omega}_n/\tilde{\Gamma}$ , using (3.23), Lemma 6.1 in the Appendix and similar arguments as in [16] (see also [20] for more details), we have that

$$\iint_{\tilde{\Omega}_n/\tilde{\Gamma}} \mathcal{K}_n^2(x, y) dx dy = \mathcal{O}(\log r_n). \quad (5.6)$$

Consequently, plugging (5.5) and (5.6) into (5.4) we obtain (5.3).  $\square$

**Proof of Theorem 1.2.** Thanks to Proposition 5.1, Proposition 5.2 and Soshnikov's central limit theorem, the proof follows analogously as that of Theorem 1.1. The key ingredient for the computations of  $\Lambda_{i,j}$ , as in the proof of Proposition 5.2, is that for any given set  $\Lambda$  in the neighborhood of  $(t, t)$  with  $t \rightarrow 1^-$ ,  $n(1-t)^{\frac{3}{2}} \rightarrow \infty$ ,

$$\iint_{\Lambda} \mathcal{K}_n^2(x, y) = \frac{1}{2\pi^2} \iint_{\Lambda \cap \tilde{\Gamma}} \frac{1}{(x-y)^2} dx dy + \mathcal{O}(\log r_n), \quad (5.7)$$

where  $\tilde{\Gamma}$  and  $r_n$  are defined as in Proposition 5.2. We refer to [20] for more details.  $\square$

**Remark 5.3** We give below the heuristic arguments to find the suitable scaling coefficients in Theorem 1.2.

Let  $I = [t, \infty)$ . Since  $t \rightarrow 1^-$ , we set  $t = 1 - b(n)$ , where  $b(n) \rightarrow 0$  as  $n \rightarrow \infty$  and will be chosen later. By Proposition 5.1 and Proposition 5.2,

$$\mathbb{P}(x_{n-k} < t) = \mathbb{P}(\#I \leq k) = \mathbb{P}\left(\frac{\#I - \mathbb{E}(\#I)}{\sqrt{\text{Var}(\#I)}} \leq \frac{k - a_2 n b^{\frac{3}{2}}(n)}{a_3 \sqrt{\log(n b^{\frac{3}{2}}(n))}} + o(1)\right), \quad (5.8)$$

where  $a_2$  is defined as in Theorem 1.2 (i), and  $a_3 = \frac{1}{\sqrt{2\pi^2}}$ . Since the order of the denominator is  $\sqrt{\log n}$ , in order to apply Soshnikov's central limit theorem,  $k$  in the numerator shall be canceled with  $a_2 n b^{\frac{3}{2}}(n)$ , and the remaining term shall be  $a_3 \sqrt{\log(n b^{\frac{3}{2}}(n))} \cdot \xi$  with  $\xi$  fixed.

For this purpose, we set  $b(n) = [\frac{k}{a_2 n}(1+c)]^{\frac{2}{3}}$ , and by direct computations we infer that

$$c = -\frac{a_3 \sqrt{\log k}}{k} \cdot \xi.$$

Then, taking Taylor's expansion shows that

$$\begin{aligned} 1 - t &= b(n) \\ &= \left[ \frac{k}{a_2 n} \left( 1 - \frac{a_3 \sqrt{\log k}}{k} \cdot \xi \right) \right]^{\frac{2}{3}} \\ &= \left( \frac{k}{a_2 n} \right)^{\frac{2}{3}} \left[ 1 - \frac{2}{3} \frac{a_3 \sqrt{\log k}}{k} \cdot \xi + \mathcal{O}\left( \frac{\log k}{k^2} \cdot \xi^2 \right) \right] \end{aligned}$$

Therefore, we can take

$$\begin{aligned} t &= 1 - \left(\frac{k}{a_2 n}\right)^{\frac{2}{3}} \left[1 - \frac{2}{3} \frac{a_3 \sqrt{\log k}}{k} \cdot \xi\right] \\ &= 1 - \left(\frac{k}{a_2 n}\right)^{\frac{2}{3}} + \frac{2a_3}{3a_2^{\frac{2}{3}}} \frac{\sqrt{\log k}}{n^{\frac{2}{3}} k^{\frac{1}{3}}} \cdot \xi, \end{aligned}$$

which implies the scaling coefficients in Theorem 1.2.

## 6 Appendix

**Proof of (3.11).** First suppose that  $x, y \in (1 - \delta, 1)$ . As in the case that  $x, y \in (-1 + \delta, 1 - \delta)$ , (3.7) holds, i.e.,

$$2\pi i(x - y)\mathcal{K}_n(x, y) = (e^{-n\varphi_n(x)}, e^{n\varphi_n(x)})S(x)^T S(y)^{-T} (-e^{n\varphi_n(y)}, e^{-n\varphi_n(y)})^T. \quad (6.1)$$

To calculate explicitly the right hand side above, we note that for  $x \in (1 - \delta, 1)$ ,  $f_n(x + i\epsilon)$  lies in the region *II* in (2.27), then as  $\epsilon \rightarrow 0$ ,

$$\Psi^\sigma(f_n(x)) = [AI(f_n(x))]e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

which by (2.29) and (2.26) yields that

$$S = RE_n[AI(f_n)]e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} e^{n\varphi_n\sigma_3}. \quad (6.2)$$

Consequently, plugging (6.2) into (6.1), straightforward computations yield (3.11) for  $x, y \in (1 - \delta, 1)$ .

Similarly, when  $x, y \in (1, 1 + \delta)$ , by (2.18), (2.19), (3.5) and (2.17),

$$U = e^{n\frac{ln}{2}\sigma_3} S e^{n(g_n - \frac{ln}{2})\sigma_3},$$

and

$$2\pi i(x - y)\mathcal{K}_n(x, y) = (e^{-n\varphi_n(x)}, 0)S^T(x)S^{-T}(y)(0, e^{-n\varphi_n(y)})^T. \quad (6.3)$$

Since for  $x \in (1, 1 + \delta)$ ,  $f_n(x + i\epsilon)$  is in the region *I* in (2.27), we then deduce from (2.29) and (2.26) that

$$S = RE_n[AI(f_n)]e^{-\frac{\pi i}{6}\sigma_3} e^{n\varphi_n\sigma_3}. \quad (6.4)$$

Combining (6.3) and (6.4) yields directly (3.11) for  $x, y \in (1, 1 + \delta)$ , thereby completing the proof of (3.11).  $\square$

**Proof of (3.14).** We first show that  $I_2(x, y)$  is of order  $n^{-\frac{5}{6}}$ . Indeed, by the expressions of  $AI$ ,  $E_n$  and  $\det[AI(z)] = \frac{-1}{2\pi i} e^{-\frac{\pi i}{3}}$  (cf. p.890 in [3]), we derive from (3.13) that

$$\begin{aligned} I_2(x, y) &= -2\pi i e^{\frac{\pi i}{3}} (H_n(x)Ai(f_n(x)), H_n^{-1}(x)Ai'(f_n(x))) \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}^T \\ &\quad \Delta_R(x, y)R^{-T}(y) \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}^{-T} (-H_n^{-1}(y)Ai'(f_n(y)), H_n(y)Ai(f_n(y)))^T. \end{aligned}$$

Note that, by  $H_n = f_n^{\frac{1}{4}} a^{-1}$  and (2.22),

$$H_n = n^{\frac{1}{6}}(x-1)^{\frac{1}{4}}(\widehat{\phi}_n)^{\frac{1}{6}}\left(\frac{x-1}{x+1}\right)^{-\frac{1}{4}} = n^{\frac{1}{6}}(x+1)^{\frac{1}{4}}(\widehat{\phi}_n)^{\frac{1}{6}} = \mathcal{O}(n^{\frac{1}{6}}).$$

Moreover, for  $x \in \mathbb{R}$ ,  $|Ai(x)| = \mathcal{O}(1)$  and  $|Ai'(f_n(x))| = \mathcal{O}(|f_n|^{\frac{1}{4}}) = \mathcal{O}(n^{\frac{1}{6}})$ . But by (2.32) and (2.33),  $\Delta R(x, y)R^{-T}(y) = \mathcal{O}(n^{-1})$ . Thus  $I_2(x, y)$  is of order  $\mathcal{O}(n^{\frac{1}{6}})\mathcal{O}(n^{-1}) = \mathcal{O}(n^{-\frac{5}{6}})$ .

It remains to check the first term in the right hand side of (3.12). Indeed, it follows from (3.12) and similar computation as above that

$$\begin{aligned} & e^{-\frac{\pi i}{3}}(1, 0)[AI(f_n(x))]^T E_n^T(x) E_n^{-T}(y)[AI(f_n(y))]^{-T}(0, 1)^T \\ &= (-2\pi i) \left[ -Ai(f_n(x))Ai'(f_n(y)) \frac{f_n^{\frac{1}{4}}(x)}{f_n^{\frac{1}{4}}(y)} \frac{a(y)}{a(x)} + Ai'(f_n(x))Ai(f_n(y)) \frac{f_n^{\frac{1}{4}}(y)}{f_n^{\frac{1}{4}}(x)} \frac{a(x)}{a(y)} \right]. \end{aligned}$$

which yields the desired term in (3.14).  $\square$

**Proof of (3.15).** The proofs are similar as those of (3.11). First suppose that  $x, y \in (-1, -1 + \delta)$ . As in the case that  $x, y \in (1 - \delta, 1)$ ,

$$2\pi i(x - y)\mathcal{K}_n(x, y) = (e^{-n\varphi_n(x)}, e^{n\varphi_n(x)})S(x)^T S(y)^{-T}(-e^{n\varphi_n(y)}, e^{-n\varphi_n(y)})^T. \quad (6.5)$$

Note that for  $x \in (-1, -1 + \delta)$ ,  $-\widetilde{f}_n(x + i\epsilon)$  lies in the region *III* in (2.27). Then, as  $\epsilon \rightarrow 0$ ,

$$\Psi^\sigma(-\widetilde{f}_n(x)) = [\widetilde{AI}(-\widetilde{f}_n(x))]e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

which by (2.29) and (2.28) yields that

$$S = R\widetilde{E}_n\sigma_3[\widetilde{AI}(-\widetilde{f}_n)]e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \sigma_3 e^{n\widetilde{\varphi}_n\sigma_3}. \quad (6.6)$$

Therefore, plugging (6.6) into (6.5), with  $\widetilde{\varphi}_n(z) = \varphi_n(z) + \pi i$ ,  $z \in \mathbb{C}^+$ , we obtain (3.15).

Now suppose that  $x, y \in (-1 - \delta, -1)$ . As in the case that  $x, y \in (1, 1 + \delta)$ ,

$$2\pi i(x - y)\mathcal{K}_n(x, y) = (e^{-n\varphi_n(x)}, 0)S^T(x)S^{-T}(y)(0, e^{-n\varphi_n(y)})^T. \quad (6.7)$$

Since for  $x \in (-1 - \delta, -1)$ ,  $-\widetilde{f}_n(x + i\epsilon)$  is in the region *IV* in (2.27). It then follows from (2.28) and (2.29) that, as  $\epsilon \rightarrow 0$ ,

$$S = R\widetilde{E}_n\sigma_3[\widetilde{AI}(-\widetilde{f}_n)]e^{-\frac{\pi i}{6}\sigma_3}\sigma_3 e^{n\widetilde{\varphi}_n\sigma_3}. \quad (6.8)$$

Therefore, plugging (6.8) into (6.7) yields immediately (3.15).  $\square$

**Proof of Lemma 4.1.** First, by Theorem 2.1,

$$\rho_{V_n}^{-1}(x) = 2\pi \frac{1}{\sqrt{1-x^2}} \frac{1}{h_n(x)} < 2\pi \frac{1}{\sqrt{1-(1-\delta)^2}} \frac{1}{h_0} < \infty,$$

for all  $n \geq N$  and  $x \in [-1 + \delta, 1 - \delta]$ .

As regards  $\rho'_V$ , note that

$$2\pi|\rho'_{V_n}| \leq \frac{|x|}{\sqrt{1-x^2}}|h_n(x)| + \sqrt{1-x^2}|h'_n(x)|. \quad (6.9)$$

Moreover, it follows from (2.3) and (2.5) that

$$h_n(x) = \sum_{k=0}^{m-1} 2 \frac{A_{m-k-1}}{A_m} x^{2k} + \mathcal{O}(n^{-\frac{1}{2m}}), \quad (6.10)$$

$$h'_n(x) = \sum_{k=0}^{m-1} 4k \frac{A_{m-k-1}}{A_m} x^{2k-1} + \mathcal{O}(n^{-\frac{1}{2m}}), \quad (6.11)$$

which implies that  $|h_n(x)|$  and  $|h'_n(x)|$  are uniformly bounded for all  $n \geq N$  and all  $x \in [-1 + \delta, 1 - \delta]$ . Hence, by (6.9) we complete the proof.  $\square$

**Lemma 6.1** *There exists a  $\delta_0 > 0$  such that for all  $0 < \delta \leq \delta_0$  the following holds:*

(i). *For  $x \in \mathbb{R}/(-1 - \delta, 1 + \delta)$ ,*

$$p_n(x; n) e^{-\frac{n}{2} V_n(x)} \leq C e^{-n\pi F_n(x)} \chi_{(1+\delta, \infty)}(x) + e^{-n\pi \tilde{F}_n(x)} \chi_{(-\infty, -1-\delta)}(x).$$

(ii). *For  $x \in (-1 - \delta, 1 + \delta)$ ,*

$$p_n(x; n) e^{-\frac{n}{2} V_n(x)} \leq C \left[ 1 + \frac{1}{|1-x|^{\frac{1}{4}}} \chi_{(1-\delta, 1+\delta)}(x) + \frac{1}{|1+x|^{\frac{1}{4}}} \chi_{(-1-\delta, -1+\delta)}(x) \right].$$

Here  $C$  is a constant independent of  $n$ .

**Proof.** This follows from the Plancherel-Rotach-type asymptotics for  $p_n(x; n)$  in [5] and the asymptotic estimates of the Airy functions (cf. (2.60), (2.61), (3.6) and (3.7) in [16]).  $\square$

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